INDIAN STATISTICAL INSTITUTE, CHENNAI CENTRE
Analysis-II

Final Exam **Instructor: S. Ponnusamy**

Date: 29-04-2016 **Time: 14.30-17.30** **Total Marks: 50**

Instructions:
- Write your roll number in the answer book
- Use of calculator, mobile phone and mathematical table is not allowed
- Justify your answers by clearly stating appropriate results/theorems that you use whenever required.
- $\mathbb{C}$ and $\mathbb{D}$ denote the field of complex numbers and the unit disk $|z| < 1$ in $\mathbb{C}$, respectively.
- By a domain D we mean a connected open subset $D \subseteq \mathbb{C}$.
- $\mathbb{R}$ denotes the field of real numbers.
- z denotes the complex numbers $x + iy$, where $x, y \in \mathbb{R}$.

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1. Define $d_p(x,y) = |x-y|^p$ for $x, y \in \mathbb{R}$. Determine all possible values of $p \in \mathbb{R}$ such that d_p defines a metric on $\mathbb{R}$. **4 marks**

2. If $\{a_{2n}\}_{n\geq 1}$ and $\{a_{2n+1}\}_{n\geq 1}$ are two Cauchy sequences in $\mathbb{C}$, what can be said about the convergence of $\{a_n\}_{n\geq 1}$? **2 marks**

3. Suppose that $\{a_n\}_{n\geq 1}$ is a sequence of positive real numbers such that
$$|a_{n+2} - a_{n+1}| \leq \frac{3}{n}|a_{n+1} - a_n| \ \text{ for } n \geq 1.$$
Show that $\{a_n\}$ is convergent. **4 marks**

4. Let $f(x,y) = \frac{16(x+3)}{y+2}$, where $x, y \in \mathbb{R}$ and $y \neq -2$. Determine the Taylor polynomial of f of degree 2 about the point $(1,2)$. **4 marks**

5. Consider $F(x,y) = y + xe^y - 2$ for $x, y \in \mathbb{R}$. Represent $F(x,y) = 0$ near $(2,0)$ by an implicit function ϕ. Compute $\phi'(2)$ and express $\phi'(x)$ for x near 2. **4 marks**

6. If $P(z,w)$ is a polynomial in the complex variables z and w, prove or disprove that the set $S = \{(a,b) \in \mathbb{C}^2 : P(a,b) = 0\}$ is compact. **4 marks**

7. Let α and β be two given non-zero complex numbers. Determine the region of convergence of $\sum_{n=0}^{\infty}[n^2\alpha^n - 3n(n+1)\beta^n]z^n$. **4 marks**

8. Does there exist an analytic function of z whose imaginary part is $x + y + e^x \cos y$? Justify your answer. **2 marks**

9. State and prove Maximum Modulus Theorem. **4 marks**

10. Determine all the singularities of $f(z) = \dfrac{e^z - 1}{1 - \cos z}$ in $\mathbb{C}$. If any of its singularity happens to be a pole, determine its order. Also evaluate $\displaystyle\int_{|z|=\pi} f(z)\,dz$. **4 marks**

11. State Laurent's theorem without proof. Expand $f(z) = \dfrac{3z-1}{z^2-2z-3}$ in Laurent series valid for $1 < |z| < 3$. **4 marks**

12. Evaluate the Cauchy principal value of $\displaystyle\int_0^\infty \frac{x^2}{x^4+x^2+1}\,dx$ **4 marks**

13. State the following theorems without proof. **6 marks**

    (a) Cantor intersection theorem

    (b) Inverse function theorem of functions of several variables

    (c) Taylor's theorem of functions of several variables

    (d) Schwarz' lemma

Bonus Mark Questions:

14. If $f = u + iv$ is an entire function such that $u - v = e^x(\cos y - \sin y)$, find $f(z)$ in terms of z. **3 marks**

15. Characterize all entire functions $f(z)$ such that $|f(x+iy)| \le 10e^x$ for all real x, y. **2 marks**

16. Let $\Omega = \{(x,y) \in \mathbb{R}^2 : x + y \neq -1\}$ and let $f : \Omega \to \mathbb{R}^2$ be defined by

$$f(x,y) = \left(\frac{x}{1+x+y}, \frac{y}{1+x+y}\right).$$

Show that f is one-to-one and determine the formula for f^{-1} and calculate Jacobian of f^{-1}. **5 marks**